Conformal Decomposition of Integral Tensions and Potentials of Signed Graphs

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Abstract

Given a subgroup Γ of an integral chain group over a set E. A nonzero chain g of Γ is said to be *conformally decomposable* if there exist nonzero chains g_1, g_2 of Γ such that $g = g_1 + g_2$ and $g_1(e)g_2(e) \geq 0$ for all $e \in E$. For a signed graph Σ with edge set E, there are two subgroups $F(\Sigma, \mathbb{Z})$ and $T(\Sigma, \mathbb{Z})$ of the 1-chain group $C_1(\Sigma, \mathbb{Z})$, known as the flow lattice and tension lattice of Σ. The conformally indecomposable flows of $F(\Sigma, \mathbb{Z})$ are classified in [5, 6] as signed-graphic circuit flows and a class of characteristic vectors of certain directed Eulerian cycle-trees. In this paper we classify conformally indecomposable tensions of $T(\Sigma, \mathbb{Z})$ as characteristic vectors of signed-graphic directed bonds and a class of characteristic vectors of directed semi-bonds and directed hyper-bonds. The half-spin structures ($\pm\frac{1}{2}$-potential functions) of Σ correspond to characteristic vectors of directed hyper-bonds. A byproduct is the classification of conformally indecomposable integral potentials.

1 Introduction

A *signed graph* is a graph whose edges are labeled with either a positive sign or a negative sign. Zaslavsky [13, 14, 15] introduced two matroids for a signed graph by extending the graphic notions of circuit, bond, and orientation to signed graphs, and notions of directed circuit, directed bond, and Laplacian to directed signed graphs. Chen and Wang [4], based on the work of Zaslavsky, introduced flow (tension) lattices (spaces) of signed graphs and obtained fundamental properties on flows and tensions. To understand how integral flows are constructed from more basic flows, Chen and Wang [5] and Chen, Wang, and Zaslavsky [6] further introduced conformal decomposition of integral flows, and classified conformally indecomposable integral flows, using algorithmic method and resolution to double covering graph respectively. In this paper we introduce conformal decomposition of integral tensions and classify conformally indecomposable integral tensions and conformally indecomposable integral potential functions.

For unsigned graphs it is easy to see that conformally indecomposable integral tensions are simply graphic bond vectors. For signed graphs, however, we shall see that conformally indecomposable integral tensions are much richer than that of unsigned graphs. In fact, in addition to reduced characteristic vectors of directed bonds, the fixed spin (signs on edges) produces a new class of characteristic vectors of so-called *directed semi-bonds* and *directed hyper-bonds*,

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which are not conformally decomposable at integer scale whereas conformally decomposable at half-integer scale into characteristic vector of signed-graphic directed bonds. The similar phenomenon of half-integer had been happened in the work of Geelen and Guenin [8] in packing odd circuits in Eulerian graphs. If one thinks of reduced characteristic vectors of directed bonds to be at atomic level, then conformally indecomposable integral tensions may be viewed to be at molecular level.

Given a signed graph $\Sigma = (V, E, \sigma)$ throughout, that is, (V, E) is an ordinary finite graph with possible loops and multiple edges, V is the vertex set, E is the edge set, and $\sigma : E \to \{-1, 1\}$ is the *sign function*. Each edge e is incident with exact two vertices u, v, called the *end-vertices* of e, written $e = uv$ or $\mathrm{End}(e) = \{u, v\}$; if $u = v$, e is known as a *loop* and $\mathrm{End}(e) = \{u, u\}$ is a multiset. A *link* is a non-loop edge. We denote by $E^+(\Sigma)$ the set of positive edges of Σ and by $E^-(\Sigma)$ the set of negative edges. For signed subgraphs Σ_i of Σ with vertex sets X_i, $i = 1, 2$, we denote by $[X_1, X_2]$ or $[\Sigma_1, \Sigma_2]$ the set of edges between vertices of Σ_1 and vertices of Σ_2.

Every edge subset $F \subset E$ induces a signed subgraph $\Sigma(F) := (V(F), F, \sigma|_F)$, where $V(F)$ consists of end-vertices of edges in F. Every vertex subset $X \subset V$ induces a signed subgraph $\Sigma(X) := (X, E(X), \sigma|_{E(X)})$, where $E(X)$ is the set of edges having end-vertices in X. A *cycle* of Σ is a simple closed path. The *sign* of a cycle is the product of signs on its edges. A cycle is said to be *balanced (unbalanced)* if its sign is positive (negative). A signed graph is said to be *balanced* if all cycles are balanced, and *unbalanced* if one of its cycles is unbalanced. A connected component of Σ is called a *balanced (unbalanced) component* if it is balanced (unbalanced) as a signed subgraph. For undefined notions of graphs, we refer to the books [1, 2, 9]. For undefined notions of signed graphs, we refer to Zaslavsky's dynamic survey [16].

A *circuit* C of Σ is either (i) a balanced cycle, said to be of *Type I*; or (ii) an edge subset consisting of two unbalanced cycles C_1, C_2, written $C = C_1C_2$ and said to be of *Type II*, such that $V(C_1) \cap V(C_2)$ contains exactly one vertex; or (iii) an edge subset consisting of two unbalanced cycles C_1, C_2, and a simple path P (called *circuit path*) containing at least one edge, written $C = C_1PC_2$ and said to be of *Type III*, such that $V(C_1) \cap V(C_2) = \varnothing$ and $\big(V(C_1) \cup V(C_2)\big) \cap V(P)$ contains exactly the initial and terminal vertices of P.

An *orientation* on an edge $e = uv$ is an assignment of two arrows at its end-vertices u, v such that the two arrows are in the same direction if $\sigma(e) = 1$ and in opposite directions if $\sigma(e) = -1$. An edge with an orientation is known as an *oriented edge*. If e is positive, an oriented edge $\vec{e}$ may be written as $\vec{u}\vec{v}$ or $\overleftarrow{v}\overleftarrow{u}$ if its two arrows point from u to v. If e is negative, an oriented edge $\vec{e}$ may be written as $\overleftarrow{u}\vec{v}$ if its two arrows point outward, and be written as $\vec{u}\overleftarrow{v}$ if its two arrows point inward. Every edge has exactly two orientations *opposite* each other. See Figure 1.

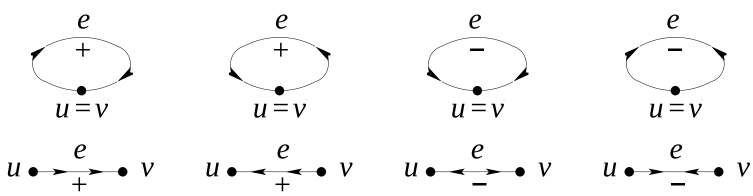

Figure 1: Orientations on loops and links.

An *orientation* on a vertex v is either a positive sign or a negative sign at the vertex; so there are exactly two oriented vertices $+v$ and $-v$. We usually write the oriented vertex $+v$ as v itself. Vertices and edges are known as 0-*cells* and 1-*cells* respectively. For c an oriented cell, we write $-c$ for the same cell with the opposite orientation. We denote by $\vec{E}$ the set of oriented edges and by $\vec{V}$ the set of oriented vertices. The sign function σ is extended to $\vec{E}$ by setting $\sigma(\pm\vec{e}) = \sigma(e)$ for $e \in E$.

An *orientation* of Σ is an assignment that each edge is given one of its two orientations, and can be viewed as a subset $\omega \subset \vec{E}$ satisfying $\omega \cap (-\omega) = \varnothing$ and $\omega \cup (-\omega) = \vec{E}$, where $-\omega = \{-e : e \in \omega\}$. Signed graph Σ with an orientation ω is considered as a bidirected ordinary graph by Edmonds [7], Bouchet [3], and Khelladi [10]; we call it an *oriented signed* *graph*, denoted (Σ, ω). We may encode arrows of an oriented edge at its end-vertices by signs $\pm$, so that an orientation may be locally described by an incidence function on $\vec{V} \times \vec{E}$.

An *incidence function* of Σ is a multivalued pairing $[\,,\,] : \vec{V} \times \vec{E} \to \{-1, 0, 1\}$ such that

$$[-u, e] = [u, e], \quad [u, -e] = -[u, e],$$

where $u \in \vec{V}, e \in \vec{E}$, and satisfy the following properties:

(1) $[u, e] = 0$ if u is not an end-vertex of e,

(2) $[u, e] = +1$ if e is a link at u and its arrow points towards u,

(3) $[u, e] = -1$ if e is a link at u and its arrow points away from u,

(4) $[u, e] = \{+1, -1\}$ if e is a positive loop at u,

(5) $[u, e] = \{+1, +1\}$ if e is a negative loop at u and its arrows point towards u,

(6) $[u, e] = \{-1, -1\}$ if e is a negative loop at u and its arrows point away from u.

It is understood that $[u, e]$, whenever e is an oriented loop at a vertex u, equals one of its two given values. For instance, for $e = uv$ either a loop or a link, we always have

$$\sum_{w \in \mathrm{End}(e)} [w, e] = [u, e][v, e] = -\sigma(e). \tag{1.1}$$

An *i-chain* f of Σ, with coefficients in an abelian group A, is a function from the set of oriented i-cells to A, satisfying $f(-c) = -f(c)$ for all oriented i-cells c, where $i = 0, 1$. A 0-chain is known as a *potential function* of Σ. Let $C_i(\Sigma, A)$ denote the group of all i-chains of Σ with coefficients in A. The *boundary operator* $\partial : C_1(\Sigma, A) \to C_0(\Sigma, A)$ is a group homomorphism defined by

$$\partial e = \begin{cases} v - u & \text{if } e = \vec{u}\vec{v}, \\ u + v & \text{if } e = \overleftarrow{u}\vec{v}. \end{cases} \tag{1.2}$$

The *co-boundary operator* $\delta : C_0(\Sigma, A) \to C_1(\Sigma, A)$ is the adjoint of ∂, given by

$$(\delta p)(e) = \begin{cases} p(v) - p(u) & \text{if } e = \vec{u}\vec{v}, \\ p(u) + p(v) & \text{if } e = \overleftarrow{u}\vec{v}. \end{cases} \tag{1.3}$$

The *flow group* of Σ is the chain subgroup $F(\Sigma, A) := \ker \partial$, whose elements are called *flows* with values in A or just *A-flows*. The vector space $F(\Sigma, \mathbb{R})$ is known as the *flow space* and $F(\Sigma, \mathbb{Z})$ the *flow lattice*.

A *direction* of a circuit C is an orientation ω_C on the signed subgraph $\Sigma(C)$ such that $(\Sigma(C), \omega_C)$ has neither a source nor a sink. There are exactly two directions $\pm\omega_C$ (opposite each other) on C, and $(C, \pm\omega_C)$ are called *directed circuits*. A *circuit vector* of Σ, associated with a directed circuit (C, ω_C), is a chain $I_{(C,\,\omega_C)} : \vec{E} \to \mathbb{Z}$ defined by

$$I_{(C,\,\omega_C)}(e) = \begin{cases} 2 & \text{if } e \in \omega_C \text{ and is on the circuit path}, \\ 1 & \text{if } e \in \omega_C \text{ and is not on the circuit path}, \\ 0 & \text{if } e \notin \omega_C \cup (-\omega_C). \end{cases} \tag{1.4}$$

It is easy to see that $\partial I_{(C,\,\omega_C)} = 0$; so $I_{(C,\,\omega_C)}$ is a flow.

A *tension* with values in A or just *A-tension* is a chain $g \in C_1(\Sigma, A)$ such that for each directed circuit (C, ω_C) of Σ,

$$\sum_{e \in \omega_C} I_{(C,\,\omega_C)}(e) g(e) = 0. \tag{1.5}$$

The *tension group* $T(\Sigma, A)$ with coefficients in A is the group of all A-tensions of Σ. The vector space $T(\Sigma, \mathbb{R})$ is known as the *tension space* and $T(\Sigma, \mathbb{Z})$ the *tension lattice*.

A *cut* of Σ is a nonempty edge subset of the form $U = [X, X^c] \cup E_X$, where $X \subset V$ is a nonempty vertex subset, $X^c := V \smallsetminus X$ is the complement of X, and $E_X \subset E(X)$ is a minimal edge subset to have the signed subgraph $\Sigma(X) \smallsetminus E_X$ balanced, that is, $(\Sigma(X) \smallsetminus E_X) \cup \{e\}$ is unbalanced for each $e \in E_X$ whenever $E_X \neq \varnothing$. A cut $U = [X, X^c] \cup E_X$ is said to be of *Type I* if $E_X = \varnothing$, and of *Type II* if $X = V$, and of *Type III* if both E_X and $[X, X^c]$ are nonempty. A cut $[X, X^c] \cup E_X$ is called a *uni-cut* if $\Sigma(X)$ is connected. A *bond* is a minimal cut in the sense that it does not contain properly any cut.

A *semi-bond* is a uni-cut $U = [X, X^c] \cup E_X$ such that $\Sigma(X^c)$ contains some balanced components Σ_0 that connect to $\Sigma(X)$, and $\Sigma(X \cup V(\Sigma_0)) \smallsetminus E_X$ is unbalanced for each such Σ_0. Bond and semi-bond are initially called *minimal directed cut* by Chen and Wang [4, p. 268], since such a cut with a direction does not properly contain any directed cut. Notice that every cut is a disjoint union of bonds and semi-bonds; see Chen and Wang [4, Theorem 2.5].

A *switching* is a sign function $\nu : V \to \{1, -1\}$. A switching function ν transforms $\Sigma = (V, E, \sigma)$ with an orientation ω into another signed graph $\Sigma^\nu = (V, E, \sigma^\nu)$ with orientation ω^ν, where

$$\sigma^\nu(e) := \nu(u)\sigma(e)\nu(v), \quad e = uv \in E, \tag{1.6}$$

$$\omega^\nu := \{e^\nu \mid e \in \omega\}, \qquad [u, e^\nu] = \nu(u)[u, e], \ u \in \mathrm{End}(e). \tag{1.7}$$

Indeed, $[u, e^\nu][v, e^\nu] = -\sigma^\nu(e)$. Switching preserves balance (of cycles and components), directed walks, circuits, cuts, bonds, semi-bonds, hyper-bonds, and bilinear form (see below), etc.

A *direction* of a cut $U = [X, X^c] \cup E_X$ is an orientation ω_U on the signed subgraph $\Sigma(U)$ such that there exists a switching function ν_X, satisfying

(i) $\nu_X(X^c) = 1$, $\sigma^\nu(E_X) = -1$, $\sigma^\nu(E(X) \smallsetminus E_X) = 1$, and

(ii) $[u, e^\nu] = 1$ for all $e \in \omega_U$ with end-vertices $u \in X$.

It is easy to see that there exist exactly two opposite directions $\pm\omega_U$ on each cut U, and $(U, \pm\omega_U)$ are called *directed cuts*.

A *cut vector* of Σ, associated with a directed cut (U, ω_U), is a chain $I_{(U,\,\omega_U)} : \vec{E} \to \mathbb{Z}$ defined by

$$I_{(U,\,\omega_U)}(e) = \begin{cases} 1 & \text{if } e \in \omega_U|_{[X,X^c]}, \\ 2 & \text{if } e \in \omega_U|_{E_X}, \\ 0 & \text{if } e \notin \omega_U \cup (-\omega_U). \end{cases} \tag{1.8}$$

We shall see that $I_{(U,\,\omega_U)}$ is a tension of Σ. A cut vector is called a *bond vector* (*semi-bond vector*) if the cut is a bond (semi-bond). If U is a bond of Type II, the chain

$$\tilde{I}_{(U,\,\omega_U)} := \frac{1}{2} I_{(U,\,\omega_U)} \tag{1.9}$$

is called a *reduced bond vector*. Bond vectors of Types I and III are already reduced.

A *hyper-bond* of Σ is an edge subset $U \subset E$ of the form

$$U = \bigcup_{1 \le i < j \le m} [X_i, X_j] \cup \bigcup_{k=1}^{m} E_{X_k}, \tag{1.10}$$

where $\{X_1, \ldots, X_m\}$ is a vertex partition of a component of Σ and $E_{X_k} \subset E(X_k)$ are edge subsets, such that all $E_{X_k} \neq \varnothing$ and $(E(X) \smallsetminus E_{X_k}) \cup \{e\}$ is unbalanced for each edge $e \in E_{X_k}$. A *direction* of a hyper-bond U is an orientation ω_U on $\Sigma(U)$ such that the orientations of edges in ω_U^ν have their arrows all pointing outward or all pointing inward, where ν is a switching such that $\sigma^\nu|_U = -1$ and $\sigma^\nu|_{U^c} = 1$. It is easy to see that there exist exactly two opposite directions $\pm\omega_U$ on each hyper-bond U, and $(U, \pm\omega_U)$ are called *directed hyper-bonds*.

A *hyper-bond vector* is a 1-chain given by the characteristic function of a direction ω_U of a hyper-bond U, that is, $I_{(U,\,\omega_U)}(e) = 1$ for $e \in \omega_U$ and $I_{(U,\,\omega_U)}(e) = 0$ for $e \notin \omega_U \cup (-\omega_U)$. If $m = 1$, the hyper-bond is just a bond of Type II and its hyper-bond vector is a reduced bond vector of Type II.

The *circuit lattice* $Z(\Sigma, \mathbb{Z})$ is the $\mathbb{Z}$-span of circuit vectors of Σ; the *bond lattice* $B(\Sigma, \mathbb{Z})$ is the $\mathbb{Z}$-span of bond vectors of Σ. The *reduced bond lattice* $\tilde{B}(\Sigma, \mathbb{Z})$ is the $\mathbb{Z}$-span of reduced bond vectors of Σ. Let R be a commutative ring with unity 1. The chain group $C_1(\Sigma, R)$ is an algebra with a canonical R-bilinear form

$$\langle f, g\rangle := \sum_{\vec{e}\in\omega} f(\vec{e})g(\vec{e}) = \sum_{e\in E} f(e)g(e), \tag{1.11}$$

where ω is an orientation of Σ. Here we can simply write $f(\vec{e})g(\vec{e})$ as $f(e)g(e)$ because it does not matter which orientation $\vec{e}$ is selected for the edge e in the sum. It is known that the vector spaces $F(\Sigma, \mathbb{R})$ and $T(\Sigma, \mathbb{R})$ are orthogonal complements of each other in the Euclidean space $C_1(\Sigma, \mathbb{R}) \cong \mathbb{R}^E$; see Chen and Wang [4, Theorem 4.7].

Given a chain subgroup $\Gamma \subset C_1(\Sigma, \mathbb{Z})$. The *support* of a chain $f \in C_1(\Sigma, \mathbb{Z})$, denoted $\operatorname{supp} f$, is an edge subset consisting of edges e such that $f(\vec{e}) \neq 0$, where $\vec{e}$ is e with an orientation. Recall that a nonzero chain $f \in \Gamma$ is said to be *primitive* if it is not an integral multiple of any chain in Γ other than $\pm f$. Following Tutte [11, 12], we call a chain f to be *elementary* if it is primitive and its support is minimal, that is, there is no nonzero chain $g \in \Gamma$ such that $\operatorname{supp} g$ is properly contained in $\operatorname{supp} f$. Tutte's definition of elementary chain is slightly different from ours, since Tutte did not require elementary chains to be primitive. It is well-known and easy to see that the collection of supports of elementary chains of Γ forms a circuit system of a matroid on the edge set $E(\Sigma)$, called the *matroid* of the chain group Γ, denoted $M(\Gamma)$.

A nonzero chain $f \in \Gamma$ is said be *conformally decomposable* if there exist nonzero chains $f_1, f_2 \in \Gamma$ such that $f = f_1 + f_2$ and $f_1(\vec{e})f_2(\vec{e}) \geq 0$ for all $e \in E$. This means that if $f_1(\vec{e}), f_2(\vec{e})$ are nonzero then both $f_1(\vec{e})$, $f_2(\vec{e})$ are positive or both are negative. We also say that f_i *conforms to the sign pattern* of f. A nonzero chain of Γ is said to be *conformally indecomposable* if it is not conformally decomposable. The conformally indecomposable integral flows are classified as characteristic vectors of so-called directed Eulerian cycle-trees by Chen and Wang [5] by algorithmic method, and by Chen, Wang, and Zaslavsky [6] by resolution to a double covering graph. The present paper is to classify conformally indecomposable integral tensions.

Main Theorem (Classification of Conformally Indecomposable Integral Tensions)**.** *An integral tension is conformally indecomposable if and only if it is either a bond vector of Type I or Type III, or a semi-bond vector, or a hyper-bond vector. (Reduced bond vector of Type II is a special hyper-bond vector).*

Semi-bond vectors and hyper-bond vectors can be further decomposed conformally into reduced bond vectors at half-integer scale, but cannot be so decomposed conformally at integer scale. However, they can be further decomposed non-conformally at integer scale into reduced bond vectors. Figure 2 demonstrates a directed hyper-bond and its hyper-bond vector and $\frac{1}{2}$-potential function. Figure 3 demonstrates a general pattern of a semi-bond decomposed into two bonds of Types II or III.

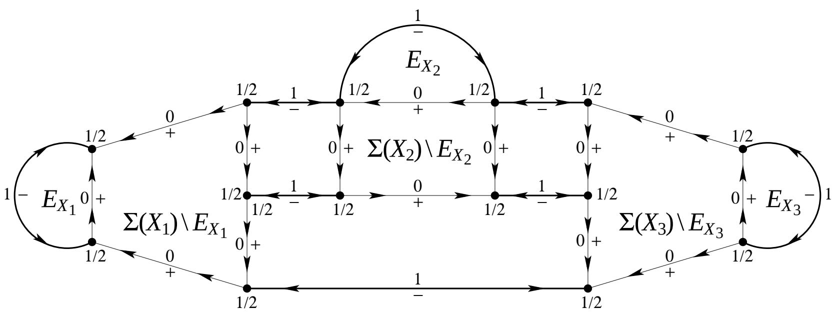

Figure 2: A hyper-bond vector and its $\frac{1}{2}$-potential function

There is a matroid $M(\Sigma)$ associated with Σ, introduced by Zaslavsky [13, 15] and is called the *signed-graphic matroid* (or *frame matroid*) of Σ, whose circuit system consists of the circuits defined above. Let $M^*(\Sigma)$ denote the dual matroid of $M(\Sigma)$; its circuits consist of the bonds defined above, called the *bond matroid* of Σ. It is anticipated that the signed-graphic matroid is the same matroid of the flow lattice, that is, $M(\Sigma) = M(F(\Sigma, \mathbb{Z}))$. However, it seems that the fact was never stated but was assumed without argument by Bouchet [3] and Khelladi [10], until it is clarified recently by Chen and Wang [5] as a by-product of the classification of conformally indecomposable integral flows. Here we further confirm that the signed-graphic bond matroid is the same matroid of the tension lattice.

Corollary 1 (Characterization of Signed-Graphic Bonds)**.** *The bond matroid of a signed graph Σ is the same matroid of its tension lattice, that is, $M^*(\Sigma) = M(T(\Sigma, \mathbb{Z}))$.*

A nonzero integral potential $p \in C_0(\Sigma, \mathbb{Z})$ is said to be *conformally decomposable* if there exist nonzero integral potentials p_1, p_2 such that $p = p_1 + p_2$, $p_1(v)p_2(v) \geq 0$ for all $v \in V$, and $\delta p = \delta p_1 + \delta p_2$ is a conformal decomposition in $T(\Sigma, \mathbb{Z})$. The following corollary characterizes conformally indecomposable integral potentials.

Corollary 2 (Characterization of Conformally Indecomposable Integral Potentials)**.** *An integral potential function p is conformally indecomposable if and only if there exists a nonempty vertex subset $X \subset V$ such that $\Sigma(X)$ is connected, $|p(v)| = 1$ for $v \in X$ and $p(v) = 0$ for $v \notin X$.*

2 Bond, semi-bond, and hyper-bond

Associated with semi-bonds and hyper-bonds are semi-bond vectors and hyper-bond vectors, which are extra building blocks for the classification of conformally indecomposable integral tensions, in addition to reduced bond vectors. Given a cut $U = [X, X^c] \cup E_X$ of Σ. The removal of U increases the number of balanced components. If $\Sigma(X)$ is connected, so is $\Sigma(X) \smallsetminus E_X$ by definition. The following characterization of bonds is obtained by Chen and Wang [4, Proposition 2.1(c)].

Proposition 2.1 (Characterization of Bonds)**.** *Let $U = [X, X^c] \cup E_X$ be a cut of connected Σ. The following statements are equivalent.*

(a) *U is a bond of Σ.*

(b) *$\Sigma(X)$ is connected and each component of $\Sigma(X^c)$ is unbalanced.*

(c) *The removal of U increases exactly one more balanced component.*

Given a switching function ν. Let $\vec{E}(e)$ denote the set of two orientations of an edge $e = uv$. Then ν induces a bijection $\nu : \vec{E}(e) \to \vec{E}^\nu(e)$. If $\nu(u) = \nu(v)$, then $\vec{E}^\nu(e) = \vec{E}(e)$; the bijection is an identity if $\nu(u) = \nu(v) = 1$ and is a switching if $\nu(u) = \nu(v) = -1$. If $\nu(u) \neq \nu(v)$, then $\vec{E}(e)$ and $\vec{E}^\nu(e)$ are disjoint; more specifically, if $\nu(u) = 1, \nu(v) = -1$, we have

$$\vec{u}\vec{v}^{\,\nu} = \vec{u}\overleftarrow{v}, \quad \overleftarrow{u}\overleftarrow{v}^{\nu} = \overleftarrow{u}\vec{v}, \quad \overleftarrow{u}\vec{v}^{\,\nu} = \overleftarrow{u}\overleftarrow{v}, \quad \vec{u}\overleftarrow{v}^{\nu} = \vec{u}\vec{v}.$$

The switching ν induces canonical isomorphisms $\nu : C_i(\Sigma, A) \to C_i(\Sigma^\nu, A)$ $(i = 0, 1)$, defined by

$$p^\nu(u) = \nu(u)p(u), \quad u \in V; \tag{2.1}$$

$$f^\nu(e) = f(e^\nu), \qquad e \in \vec{E}^\nu, \tag{2.2}$$

where $p \in C_0(\Sigma, A)$, $f \in C_1(\Sigma, A)$. Whenever A is a commutative ring, we have

$$\langle f^\nu, g^\nu \rangle = \langle f, g \rangle. \tag{2.3}$$

Lemma 2.2. *Let ν be a switching. Then a chain $f \in C_1(\Sigma, \mathbb{Z})$ is a (conformally decomposable) flow (tension) of Σ if and only if $f^\nu \in C_1(\Sigma^\nu, \mathbb{Z})$ is a (conformally decomposable) flow (tension).*

Proof. Fix an orientation ω of Σ. Then $\omega^\nu = \{e^\nu \mid e \in \omega\}$ is an orientation of Σ^ν. If f is a flow of Σ, then for each vertex $u \in \vec{V}$,

$$\begin{aligned}(\partial f^\nu)(u) &= \sum_{x\in\omega^\nu}[u,x]f^\nu(x) \\ &= \sum_{e\in\omega}[u,e^\nu]f^\nu(e^\nu) \\ &= \sum_{e\in\omega}[\nu(u)u,e]f(e) \\ &= \nu(u)(\partial f)(u) \\ &= 0.\end{aligned}$$

This means that f^ν is a flow of Σ^ν. Given a directed circuit (C,ω_C) of Σ. Then (C^ν,ω_C^ν) is a directed circuit of Σ^ν. If f is a tension of Σ, we have

$$\sum_{x\in\omega_C^\nu} I_{(C^\nu,\,\omega_C^\nu)}(x)f^\nu(x) = \sum_{e\in\omega_C} I^\nu_{(C,\,\omega_C)}(e^\nu)f^\nu(e^\nu) = \sum_{e\in\omega_C} I_{(C,\,\omega_C)}(e)f(e) = 0.$$

This means that f^ν is a tension of Σ^ν.

The conformal decomposability follows from the induced map ν being a homomorphism. ☐

An *walk* W (*oriented walk* ω_W) on Σ is a sequence of vertices and edges (oriented edges),

$$W\ (\omega_W) = u_0e_1u_1e_2\dots u_{n-1}e_nu_n, \tag{2.4}$$

alternating between vertices and edges (oriented edges), such that $\mathrm{End}(e_i) = \{u_{i-1},u_i\}$, $i = 1,\dots,n$. The *sign* of W and ω_W is

$$\sigma(W) = \sigma(\omega_W) := \prod_{i=1}^{n}\sigma(e_i). \tag{2.5}$$

An (oriented) walk is said to be *positive* (*negative*) if its sign is positive (negative). We call W a *closed walk* if $u_n = u_0$, and call ω_W a *directed walk* or a *direction* of W if

$$[u_i,e_i] + [u_i,e_{i+1}] = 0, \quad i = 1,\dots,n-1.$$

If ω_W is a directed walk, then

$$[u_n,e_n] = -\sigma(W)[u_0,e_1]. \tag{2.6}$$

If ω_W is a closed directed walk with positive sign, then its oriented edge set forms a flow of Σ; such a flow is still denoted by ω_W.

Proposition 2.3 (Decomposition of Directed Cut and Cut Vectors)**.**

(a) *Every directed cut is a disjoint union of some directed bonds and some directed semi-bonds.*

(b) *Every cut vector can be conformally decomposed into a sum of bond vectors and semi-bond vectors.*

(c) *Every cut vector can be decomposed non-conformally into a sum of bond vectors.*

Proof. (a) and (b) Let $U = [X,X^c]\cup E_X$ be a cut with a direction ω. Let $\Sigma(X)$ be decomposed into connected components $\Sigma_1,\dots,\Sigma_m$. Then (U,ω) is a disjoint union of directed cuts (U_i,ω_i), where $U_i = [X_i,{X_i}^c]\cup E_{X_i}$, $\omega_i = \omega|_{U_i}$, $X_i = V(\Sigma_i)$, and $E_{X_i} = E(\Sigma_i)\cap E_X$. Moreover, $I_{(U,\,\omega)} = \sum_{i=1}^m I_{(U_i,\,\omega_i)}$ is a conformal decomposition.

Now we may assume that $\Sigma(X)$ is connected. Then $\Sigma(X)\smallsetminus E_X$ is connected. Let Σ_j be balanced components of $\Sigma(X^c)$ such that $[X,\Sigma_j]\neq\varnothing$ and $\Sigma_j\cup\Sigma[X,\Sigma_j]$ are balanced. Set $Y_j := V(\Sigma_j)$ and $Y := \bigcup_j X_j$. Then each $U_j := [X,Y_j]$ is a bond of Type I with direction

$\omega_j := \omega|_{U_j}$, and $U' := [X \cup Y, (X \cup Y)^c] \cup E_X$ is a cut with direction $\omega' := \omega|_{U'}$. Hence (U, ω) is a disjoint union of directed bonds (U_j, ω_j) of Type I and the directed cut (U', ω'). Moreover,

$$I_{(U,\,\omega)} = I_{(U',\,\omega')} + \sum_j I_{(U_j,\,\omega_j)}$$

is a conformal decomposition.

Furthermore, if $\Sigma((X \cup Y)^c)$ does not contain balanced component that connects to $\Sigma(X \cup Y)$, then (U', ω') is either empty or a directed bond of Types II or III. If $\Sigma((X \cup Y)^c)$ contains some balanced components Σ_k that connect to $\Sigma(X)$, then $\Sigma_k \cup \Sigma[X, \Sigma_k]$ must be unbalanced, and subsequently, (U', ω') is a directed semi-bond.

(c) It follows from Proposition 2.4(b) that each semi-bond vector can be decomposed non-conformally into a sum of bond vectors. □

Semi-bond can be characterized as a uni-cut whose removal increases at least two balanced components and for each pair Σ_1, Σ_2 of increased balanced components, $\Sigma_1 \cup \Sigma_2 \cup [\Sigma_1, \Sigma_2]$ is unbalanced if $[\Sigma_1, \Sigma_2] \neq \varnothing$.

Proposition 2.4 (Decomposition of Semi-Bond Vectors)**.** *Let $U = [X, X^c] \cup E_X$ be a semi-bond of connected Σ with a direction ω. Let $\Sigma_1, \ldots, \Sigma_m$ be balanced components of $\Sigma(X^c)$. Set $Y_i := V(\Sigma_i)$ and $Y := \bigcup_{i=1}^m Y_i$. Let ν be a switching such that $\sigma^\nu|_{E_X} = -1$, $\sigma^\nu|_{E(X) \cup E(Y) \smallsetminus E_X} = 1$, and $[u, e^\nu] = 1$ for $e \in \omega$ with end-vertices $u \in X$. We then have*

(a) Conformal Decomposition at Half-Integer Scale: *Each $U'_j := [X \cup Y, (X \cup Y)^c] \cup E_{X \cup Y,\, j}$ is a bond of Types II or III of Σ with a direction $\omega'_j := \omega|_{U_j}$, $j = 1, 2$, where*

$$E_{X\cup Y,\,1} := E_X \cup E^-\big(\Sigma^\nu[X, Y]\big), \quad E_{X\cup Y,\,2} := E_X \cup E^+\big(\Sigma^\nu[X, Y]\big);$$

and $I_{(U,\,\omega)}$ is decomposed conformally into two half bond vectors

$$I_{(U,\,\omega)} = \frac{1}{2}\big[I_{(U'_1,\,\omega'_1)} + I_{(U'_2,\,\omega'_2)}\big]. \tag{2.7}$$

In the case that $\Sigma(X^c)$ does not contain unbalanced component, the conformal decomposition is a sum of two reduced bond vectors of Type II.

(b) Non-Conformal Decomposition at Integer Scale: *Each $U_i := [Y_i, {Y_i}^c]$ is a bond of Type I of Σ with direction*

$$\omega_i := \big(\omega|_{E^-(\Sigma^\nu[Y_i, {Y_i}^c])}\big) \cup \big(-\omega|_{E^+(\Sigma^\nu[Y_i, {Y_i}^c])}\big), \quad i = 1, \ldots, m;$$

and $I_{(U,\,\omega)}$ is decomposed non-conformally into

$$I_{(U,\,\omega)} = I_{(U'_1,\,\omega'_1)} - \sum_{i=1}^m I_{(U_i,\,\omega_i)}. \tag{2.8}$$

Proof. (a) Both U'_1, U'_2 are bonds of Types II or III in Σ and overlap on $[X, Y]$. We only need to show that ω'_j are directions of U'_j, $j = 1, 2$. It is clear that ω'_1 is a direction on U'_1, since $[u, e^\nu] = 1$ for $e \in \omega'_1$ within $[X \cup Y, (X \cup Y)^c] \cup E_X$ and with end-vertices $u \in X$, and since $[v, e^\nu] = [u, e^\nu] = 1$ for $e \in \omega'_1$ within $E^-\big(\Sigma^\nu[X, Y]\big)$ and with end-vertices $u \in X, v \in Y$.

For the orientation ω'_2 on U'_2, let μ be the switching such that $\mu(Y) = -1$ and $\mu(Y^c) = 1$; consider the switching $\nu\mu$. Then $\sigma^{\nu\mu}(E_{X\cup Y,\,2}) = -1$ and $\sigma^{\nu\mu}(E(X \cup Y) \smallsetminus E_{X\cup Y,\,2}) = 1$. For $e \in \omega'_2$ within $[X \cup Y, (X \cup Y)^c] \cup E_X$ and with end-vertices $u \in X$, we have $[u, e^{\nu\mu}] = [u, e^\nu] = 1$. For $e \in \omega'_2$ within $E^+\big(\Sigma^\nu[X, Y]\big)$ and with end-vertices $u \in X, v \in Y$, we have $[u, e^{\nu\mu}] = [u, e^\nu] = 1$ and $[v, e^{\nu\mu}] = -[v, e^\nu] = [u, e^\nu] = 1$. So ω'_2 is a direction of U'_2.

Since ω'_j are restrictions of ω on U'_j, the conformal decomposition (2.7) follows.

(b) It is clear that each $U_i = [Y_i, {Y_i}^c]$ is a bond of Type I in Σ. We only need to show that each ω_i is a direction of U_i. In fact, for each $e = uv \in \omega_i$ with $u \in X, v \in Y_i$, if $e \in \omega$ within

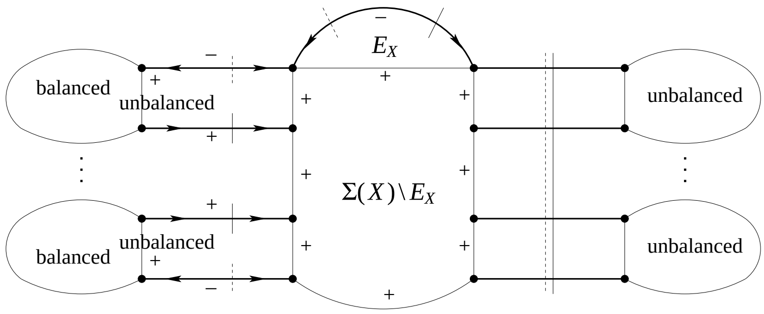

Figure 3: A semi-bond decomposed into two bonds

$E^-\big(\Sigma^\nu[Y_i, Y_i{}^c]\big)$, then $[u, e^\nu] = [v, e^\nu] = 1$; if $e \in -\omega$ within $E^+\big(\Sigma^\nu[Y_i, Y_i{}^c]\big)$, i.e., $-e \in \omega$, then $[v, e^\nu] = -[v, (-e)^\nu] = [u, (-e)^\nu] = 1$. Hence ω_i is a direction of U_i.

Notice that U_1' and U_i overlap on $E^-\big(\Sigma^\nu[Y_i, Y_i{}^c]\big)$ and their directions ω_1', ω_i agree on $E^-\big(\Sigma^\nu[Y_i, Y_i{}^c]\big)$. The directions ω, ω_i disagree on $E^+\big(\Sigma^\nu[Y_i, Y_i{}^c]\big)$. The decomposition (2.8) follows. □

Proposition 2.5 (Decomposition of Hyper-Bond Vectors). *Let* $U = \big(\bigcup_{i<j}[X_i, X_j]\big) \cup \big(\bigcup_{k=1}^m E_{X_k}\big)$ *be a hyper-bond of connected* Σ *with a direction* ω*, where* $\{X_1, \ldots, X_m\}$ *is a partition of* $V(\Sigma)$ *and* $E_{X_i} \neq \varnothing$ *for all* i*.*

(a) Conformal Decomposition at Half-Integer Scale: *If* $m = 1$*, then the hyper-bond vector is a reduced bond vector of Type II. If* $m \geq 2$*, then every* $U_k := [X_k, X_k{}^c] \cup E_{X_k}$ *is a bond of Type III with direction* $\omega_k := \omega|_{U_k}$*, and* $I_{(U,\omega)}$ *is decomposed conformally into*

$$I_{(U,\omega)} = \frac{1}{2}\sum_{i=1}^{m} I_{(U_k,\omega_k)}. \tag{2.9}$$

(b) Non-Conformal Decomposition at Integer Scale: *Let* $\tilde\Sigma$ *denote the simple graph having each* X_k *as a vertex and each nonempty edge subset* $[X_i, X_j]$ *with* $i \neq j$ *as an edge. Let* $\tilde F$ *be a minimal edge subset of* $\tilde\Sigma$ *such that* $\tilde\Sigma' := \tilde\Sigma \smallsetminus \tilde F$ *contains no odd cycle. Then* $\tilde\Sigma'$ *is bipartite with vertex bipartition* $\{\tilde X, \tilde Y\}$*,* $U' := U \cap (E(X) \cup E(Y))$ *is a bond of Type II with a direction* $\omega' := \big(\omega|_{U\cap E(X)}\big) \cup \big(-\omega|_{U\cap E(Y)}\big)$*, where*

$$X = \bigcup_{X_k \in \tilde X} X_k, \quad Y = \bigcup_{X_k \in \tilde Y} X_k,$$

and $I_{(U,\omega)}$ *is decomposed non-conformally into*

$$I_{(U,\omega)} = \sum_{X_k \subset X} I_{(U_k,\omega_k)} - \tilde I_{(U',\omega')}. \tag{2.10}$$

Proof. (a) Let ν be a switching such that $\sigma^\nu|_U = -1$, $\sigma^\nu|_{E\smallsetminus U} = 1$, and all orientation arrows for edges in ω^ν point outward or all point inward. Thus each (U_k, ω_k^ν) is a directed bond of Type III in Σ^ν if $m \geq 2$. It follows that each (U_k, ω_k) is a directed bond of Type III in Σ for $m \geq 2$. Note that for $i \neq j$, (U_i, ω_i) and (U_j, ω_j) overlap on $[X_i, X_j]$. We see that $I_{(U,\omega)} = \frac{1}{2}\sum_{k=1}^m I_{(U_k,\omega_k)}$.

(b) Let μ be a switching such that $\mu|_X = 1$, $\mu|_Y = -1$. Since $(\omega|_{U\cap E(X)})^{\nu\mu} = (\omega|_{U\cap E(X)})^\nu$, then for $e \in \omega|_{U\cap E(X)}$ with end-vertices $u, v \in X$, we have $[u, e^{\nu\mu}] = [u, e^\nu] = 1$ and $[v, e^{\nu\mu}] = [v, e^\nu] = 1$. For $e \in -\omega|_{U\cap E(Y)}$ with end-vertices $u, v \in Y$, i.e., $-e \in \omega|_{U\cap E(Y)}$, we have

$$[u, e^{\nu\mu}] = [u, -e^\nu] = [u, (-e)^\nu] = 1, \quad [v, e^{\nu\mu}] = [v, -e^\nu] = [v, (-e)^\nu] = 1.$$

Hence ω' is a direction of bond U'. Since the orientations of ω, ω' agree on $U \cap E(X)$ and disagree on $U \cap E(Y)$, we see that $I_{(U,\omega)} = \sum_{X_i \subset X} I_{(U_i,\omega_i)} - \tilde I_{(U',\omega')}$. □

The half-integer phenomenon in (2.7) and (2.9) may be related to the similar phenomenon discovered by Geelen and Guenin [8, Corollary 1.4].

Lemma 2.6. *Let* $U = [X, X^c] \cup E_X$ *be a uni-cut of connected* Σ *with a direction* ω*, and* Σ_0 *be a component of* $\Sigma(X^c)$*. Let* ν *be a switching such that* $\nu|_{X^c} = 1$*,* $\sigma^\nu|_{E_X} = -1$*, and* $\sigma^\nu|_{E(X)\smallsetminus E_X} = 1$*. Given a tension* g *of* Σ *whose support is contained in* U*.*

(a) *If* $E_X \neq \varnothing$*, then* g *is constant on* $\omega|_{E_X}$*, say,* $g = c$*.*

(b) *If* $\Sigma_0 \cup \Sigma[X, \Sigma_0]$ *is balanced, then* g *is constant on* $\omega|_{[X,\Sigma_0]}$*.*

(c) *If* Σ_0 *is balanced and* $\Sigma_0 \cup \Sigma[X, \Sigma_0]$ *is unbalanced, then* g *is constant on* $\omega_{E^+(\Sigma^\nu[X,\Sigma_0])}$*, say,* $g = a$*; and* g *is constant on* $\omega|_{E^-(\Sigma^\nu[X,\Sigma_0])}$*, say,* $g = b$*.*

(d) *If* $E_X \neq \varnothing$ *and* Σ_0 *is unbalanced, then* $g = \frac{c}{2}$ *on* $\omega|_{[X,\Sigma_0]}$*.*

(e) *If* $E_X \neq \varnothing$*,* Σ_0 *is balanced, and* $\Sigma_0 \cup \Sigma[X, \Sigma_0]$ *is unbalanced, then* $a + b = c$*.*

Proof. Since switching does not change uni-cut, we may assume $\sigma|_{E_X} = -1$ and $\sigma|_{E(X)\smallsetminus E_X} = 1$. We write $\omega_U = \{\vec{e} : e \in U\}$. Since $\Sigma(X)$ is connected, so is $\Sigma(E(X) \smallsetminus E_X)$ by definition of E_X.

(a) Given edges $e_1 = u_1v_1$ and $e_2 = u_2v_2$ in E_X. Let P_1, P_2, and P be shortest paths from v_1 to u_1, from v_2 to u_2, and from u_1 to u_2 respectively in $\Sigma(E(X) \smallsetminus E_X)$. Set $C_1 := u_1e_1P_1$ and $C_2 := u_2e_2P_2$. Then $C = C_1PC_2$ is a circuit of Type III and $W = C_1PC_2P^{-1}$ is a positive closed walk. Let ω_W be a direction of W such that the orientations of e_1 in ω_U, ω_W are the same. Then the orientations of e_2 in ω_U, ω_W must be opposite. Since $\langle \omega_W, g\rangle = 0$, it follows that $g(\vec{e}_2) = g(\vec{e}_1)$. See the left of Figure 4.

Figure 4: Cases (a) and (b)

(b) Given two edges $e_1, e_2 \in [X, \Sigma_0]$. Let C be a circuit of Type I in $\Sigma_0 \cup \Sigma[X, \Sigma_0]$, intersecting U exactly at the two edges e_1, e_2. Let ω_C be a direction of C. If the orientations of e_1 in ω_U, ω_C agree, then the orientations of e_2 in ω_U, ω_C must be opposite. Since $\langle I_{(C,\omega_C)}, g\rangle = 0$, it follows that $g(\vec{e}_2) = g(\vec{e}_1)$. See the right of Figure 4.

(c) It is analogous to (b) by letting e_1, e_2 be both positive or both negative.

(d) Given two edges $e_1 \in E_X, e_2 \in [X, \Sigma_0]$. Let $C = C_1PC_2$ be a circuit of Type III that intersects U exactly at the two edges e_1, e_2, and $C_1 \subset \Sigma(X)$, $C_2 \subset \Sigma_0$, $P = e_2$. Let ω_C be a direction of C. If the orientations of e_1 in ω_U, ω_C agree, then the orientations of e_2 in ω_U, ω_C must be opposite. Since $\langle I_{(C,\omega_C)}, g\rangle = 0$, it follows that $2g(\vec{e}_2) = g(\vec{e}_1)$. See the left of Figure 5.

Figure 5: Cases (d) and (e)

(e) Given three edges $e_1 \in E_X, e_2 \in E^+[X, \Sigma_0], e_3 \in E^-[X, \Sigma_0]$. Let C be a circuit of Type I or Type II that intersects U exactly at the three edges e_1, e_2, e_3. Let ω_C be a direction of C. It is easy to see that if the orientations of e_1 in ω_U, ω_C agree, then the orientations of e_i in ω_U, ω_C must be opposite, $i = 2, 3$. Since $\langle I_{(C,\omega_C)}, g\rangle = 0$, it follows that $g(\vec{e}_2) + g(\vec{e}_3) = g(\vec{e}_1)$. See the right of Figure 5. □

Proposition 2.7. *Let $U = [X, X^c] \cup E_X$ be a bond of Σ with a direction ω_U. Let g be a nonzero tension of Σ having support contained in U. Then g is a multiple of a reduced bond vector $I_{(U,\omega_U)}$. This means that U is a circuit of the matroid $M(T(\Sigma, \mathbb{Z})$.*

Proof. It follows from (a), (b) and (d) of Lemma 2.6. □

Proposition 2.8. *Let $U = [X, X^c] \cup E_X$ be a semi-bond of connected Σ with a direction ω_U. Let $\Sigma_1, \ldots, \Sigma_m$ be balanced components of $\Sigma(X^c)$, and $\Sigma'_1, \ldots, \Sigma'_n$ be unbalanced components of $\Sigma(X^c)$. Let ν be a switching such that $\sigma^\nu|_{E_X} = -1$, $\sigma^\nu|_{E(X)\smallsetminus E_X} = 1$, and $\sigma^\nu|_{E(\Sigma_i)} = 1$. If g is a nonzero tension of Σ having support contained in U. Then within ω_U, $g|_{E_X} = c$, $g|_{[X,\Sigma'_k]} = \frac{c}{2}$, and*

$$g|_{E^+(\Sigma^\nu[X,\Sigma_i])} = a_i, \quad g|_{E^-(\Sigma^\nu[X,\Sigma_i])} = b_i, \quad a_i + b_i = c.$$

In particular, if g is constant on $[X, \Sigma_i]$ within ω_U for each Σ_i, then $g = c\, I_{(U,\omega_U)}$.

Proof. If $E_X \neq \varnothing$, then within ω_U, $g = c$ on E_X by Lemma 2.6(a), $g = \frac{c}{2}$ on $[X, \Sigma'_k]$ by Lemma 2.6(d), $g = a_i$ on $E^+[X, \Sigma_i]$ and $g = b_i$ on $E^-[X, \Sigma_i]$ by Lemma 2.6(c), and $a_i + b_i = c$ by Lemma 2.6(e). Assume $E_X = \varnothing$. Given three edges $x_i \in E^+[X, \Sigma_i], y_i \in E^-[X, \Sigma_i], z_k \in$

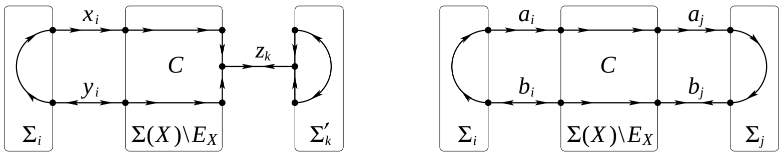

Figure 6: Left: Σ'_k exists; Right: Σ'_k does not exist

$[X, \Sigma'_k]$. Let C be a circuit of Type III of Σ that intersects U exactly at the three edges x_i, y_i, z_k. If the orientations of z_k in ω_U, ω_C agree, then the orientations of x_i, y_i in ω_U must be opposite to their orientations in ω_C. Since $\langle I_{(C,\omega_C)}, g\rangle = 0$, it follows that $g(\vec{x}_i) + g(\vec{y}_i) = 2g(\vec{z}_k)$ for $\vec{x}_i, \vec{y}_i, \vec{z}_k \in \omega_U$. Hence $a_i + b_i = c$. See the left of Figure 6.

If $E_X = \varnothing$ and all components of $\Sigma(X^c)$ are balanced, then $a_i + b_i = a_j + b_j$ for $i \neq j$. See the right of Figure 6. □

Proposition 2.9. *Let $U = \bigcup_{i<j}[X_i, X_j] \cup \bigcup_{k=1}^m E_{X_k}$ be a hyper-bond of connected Σ with a direction ω_U. Let ν be a switching such that $[u, e^\nu] = [v, e^\nu] = 1$ for $e \in \omega_U$ with end-vertices u, v. If g is a nonzero tension of Σ whose support is contained in U. Then within ω_U,*

$$g|_{E_{X_i}} = c_i, \quad g|_{[X_i,X_j]} = a_{ij}, \quad a_{ij} = \frac{1}{2}(c_i + c_j). \tag{2.11}$$

In particular, if g is constant on $\bigcup_{i=1}^m E_{X_i}$ within ω_U, then g is a multiple of $I_{(U,\omega_U)}$.

Proof. Since switching does not change hyper-bond, we may assume that $\sigma|_U = -1$, $\sigma|_{E\smallsetminus U} = 1$, and that all arrows of the orientation ω_U point outward. By Lemma 2.6(a), $g = c_i$ on E_{X_i} within ω_U. Assume $[X_i, X_j] \neq \varnothing$; then $g = a_{ij}$ on $[X_i, X_j]$ within ω_U by the proof of Lemma 2.6(b).

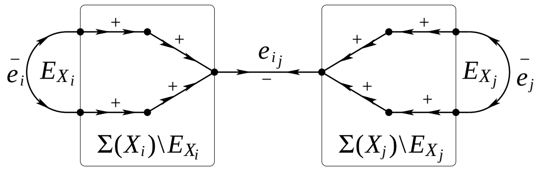

Figure 7: A circuit of Type III between two parts

Given three edges $e_i \in E_{X_i}, e_j \in E_{X_j}, e_{ij} \in [X_i, X_j]$ with $i \neq j$. Let C be a circuit of Type III that contains the three edges e_i, e_j, e_{ij} and $C \smallsetminus \{e_i, e_j, e_{ij}\}$ is contained in $(\Sigma(X_i) \smallsetminus E_{X_i}) \cup$

$(\Sigma(X_j) \smallsetminus E_{X_j})$. Let ω_C be a direction of C. If the orientations of e_{ij} in ω_U, ω_C agree, then the orientations of e_i in ω_U, ω_C are opposite, so are the orientations of e_j. Since $\langle I_{(C,\,\omega_C)}, g\rangle = 0$, it follows that $2g(e_{ij}) = g(e_i) + g(e_j)$. Hence $2a_{ij} = c_i + c_j$. See Figure 7. $\square$

It is known from Chen and Wang [4] that cut vectors are orthogonal to circuit vectors. Here we reproduce the proof of the result, using incidence numbers rather than coupling of orientations, and correct some typos in the original proof of Lemma 3.3 of [4]. It is well known that every integral flow is an integer linear combination of flows generated by directed closed positive walks. We need the following lemma to proceed.

Lemma 2.10. *Let $U = [X, X^c] \cup E_X$ be a cut of Σ with a direction ω_U. Let $W = u_0 e_1 W' e_k u_k$ be a walk with a direction ω_W, such that $e_1, e_k \in U$ and the subwalk W' (possibly empty) is contained in $E(X) \smallsetminus E_X$. Then $I_{(U,\,\omega_U)}(\vec{e}_k) = -I_{(U,\,\omega_U)}(\vec{e}_1)$, where $\vec{e}_1, \vec{e}_k \in \omega_W$.*

Proof. Write $W' = u_1 e_2 u_2 x_3 \ldots e_{k-1} u_{k-1}$. Let ν be a switching such that $\nu|_{X^c} = 1$, $\sigma^\nu|_{E_X} = -1$, $\sigma^\nu|_{E(X)\smallsetminus E_X} = 1$, and $[u, e^\nu] = 1$ for $e \in \omega_U$ with end-vertices $u \in X$. Note that W^ν is a positive walk. Then by (2.6), for $\vec{e}_1, \vec{e}_k \in \omega_W$, the sign of $I_{(U,\,\omega_U)}(\vec{e}_k)$ is

$$[u_k, \vec{e}_k^{\,\nu}] = -\sigma(W^\nu)[u_0, \vec{e}_1^{\,\nu}] = -[u_0, \vec{e}_1^{\,\nu}],$$

which is opposite to the sign of $I_{(U,\,\omega_U)}(\vec{e}_1)$. Hence $I_{(U,\,\omega_U)}(\vec{e}_k) = -I_{(U\,\omega_U)}(\vec{e}_1)$. $\square$

Proposition 2.11 (Orthogonality of Flows and Cut Vectors)**.** *Let (U, ω_U) be a directed cut of Σ, and W a closed positive walk with a direction ω_W. Then the cut vector $I_{(U,\,\omega_U)}$ is orthogonal to the flow ω_W, that is,*

$$\langle \omega_W, I_{(U,\,\omega_U)}\rangle = \sum_{e\in\omega_W} I_{(U,\,\omega_U)}(e) = 0.$$

Proof. Let $U = [X, X^c] \cup E_X$, where $X \subset V$, $E_X \subset E(X)$. Since switching preserves the bilinear form, we may assume $\sigma|_{E_X} = -1$, $\sigma|_{E(X)\smallsetminus E_X} = 1$, and $[u, e] = 1$ for $e \in \omega_U$ with end-vertices $u \in X$.

Case 1: *W is contained in $\Sigma(X^c)$.* Since U is disjoint from $\Sigma(X^c)$, it is trivial that $\langle I_{(U,\,\omega_U)}, \omega_W\rangle = 0$.

Case 2: *W is contained in $\Sigma(X)$.* Let us write $W = P_1 Q_1 P_2 Q_2 \cdots P_k Q_k$, where P_i are subwalks inside E_X and Q_i are subwalks inside $E(X) \smallsetminus E_X$. If $W \subset E_X$, then $k = 1$ and $W = P_1, Q_1 = \varnothing$. If $W \subset E(X) \smallsetminus E_X$, then $k = 1$ and $W = Q_1, P_1 = \varnothing$. Since the edges of $E(X) \smallsetminus E_X$ are positive, the edges of E_X are negative, and the walk W is positive, it follows that the sequence $P := P_1 P_2 \cdots P_k$ contains even number of edges, and $I_{(U,\,\omega_U)}$ is alternating on each ω_{P_i}. Note that $I_{(U,\,\omega_U)}$ has opposite signs at the terminal edge of ω_{P_i} and the initial edge of $\omega_{P_{i+1}}$ by Lemma 2.10. Then $I_{(U,\,\omega_U)}$ is alternating on ω_P. Hence

$$\langle \omega_W, I_{(U,\,\omega_U)}\rangle = \sum_{e\in\omega_P} I_{(U,\,\omega_U)}(e) = 0.$$

Case 3: *W passes through $[X, X^c]$.* We fix an edge $x_1 \in [X, X^c]$ with an end-vertex $u_1 \in V(X^c)$. By traveling along W, we break W into some subwalks W_j inside $E(X) \cup [X, X^c]$ and subwalks W_j' inside $E(X^c)$,

$$W = W_1 W_1' W_2 W_2' \cdots W_k W_k'.$$

It is enough to show that $\langle \omega_{W_j}, I_{(U,\,\omega_U)}\rangle = 0$ for each j. Let W_j be written as

$$W_j = u_j x_j Q_{0j} P_{1j} Q_{1j} P_{2j} Q_{2j} \cdots P_{n_j j} Q_{n_j j} v_j y_j,$$

where $x_j, y_j \in [X, X^c]$, P_{ij} are subwalks inside E_X, and Q_{ij} are subwalks inside $E(X) \smallsetminus E_X$. Likewise, $I_{(U,\,\omega_U)}$ is alternating on $\omega_{P_{ij}}$, $I_{(U,\,\omega_U)}$ has opposite signs at the terminal edge of $\omega_{P_{ij}}$ and the initial edge of $\omega_{P_{i+1,\,j}}$, $1 \le i \le n_j - 1$. Moreover, the sign of $I_{(U,\,\omega_U)}$ at the initial edge of $\omega_{P_{1j}}$ is opposite to $[u_j, x_j]$ with $x_j \in \omega_W$, and the sign of $I_{(U,\,\omega_U)}$ at the terminal edge of $\omega_{P_{n_j j}}$ is opposite to $[v_j, y_j]$ with $y_j \in \omega_W$.

Now contracting edges of Q_{ij} $(0 \le i \le n_j)$, we obtain sequences $\tilde{W}_j := u_j x_j P_j v_j y_j$, where $P_j := P_{1j} P_{2j} \cdots P_{n_j j}$. It follows from previous argument that $I_{(U,\,\omega_U)}$ is alternating on $\omega_{\tilde{W}_j}$ and

$$[v_j, y_j] = -\sigma(W_j)[u_j, x_j] \quad \text{with} \quad x_j, y_j \in \omega_W.$$

If $\sigma(W_j) = 1$, then $[v_j, y_j] = -[u_j, x_j]$ and the number of edges of P_j is even. We thus have

$$\langle \omega_{W_j}, I_{(U,\,\omega_U)} \rangle = [u_j, x_j] + [v_j, y_j] + 2 \sum_{x \in \omega_{P_j}} I_{(U,\,\omega_U)}(x) = 0.$$

If $\sigma(W_j) = -1$, then $[v_j, y_j] = [u_j, x_j]$ and the number of edges of P_j is odd. Let e be the initial edge of ω_{P_j}. We have

$$\langle \omega_{W_j}, I_{(U,\,\omega_U)} \rangle = 2[u_j, x_j] + 2I_{(U,\,\omega_U)}(e) = 0.$$

□

3 Equivalence of potentials and tensions

Recall the co-boundary operator $\delta : C_0(\Sigma, A) \to C_1(\Sigma, A)$. For each potential function $p : V \to A$, where A is an abelian group, the potential function p^ν of Σ^ν with a switching ν, defined by $p^\nu(v) = \nu(v)p(v)$ for $v \in V$, is the potential function of the tension $(\delta p)^\nu$ for Σ^ν, that is,

$$\delta p^\nu = (\delta p)^\nu. \tag{3.1}$$

In fact, let $x = uv \in \vec{\Sigma}^\nu$ and $e \in \vec{\Sigma}$ be such that $x = e^\nu$. Then

$$\begin{aligned} (\delta p^\nu)(x) &= [u, x]p^\nu(u) + [v, x]p^\nu(v) \\ &= [u, e^\nu]\nu(u)p(u) + [v, e^\nu]\nu(v)p(v) \\ &= [u, e]p(u) + [v, e]p(v) \\ &= (\delta p)(e) \\ &= (\delta p)^\nu(x). \end{aligned}$$

Fix an orientation ω of Σ. We write $\omega = \{\vec{e} : e \in E\}$. Then every 1-chain f of Σ can be viewed as a function defined on E by $f(e) = f(\vec{e})$ for $e \in E$, and every function g on E is viewed as a 1-chain of Σ by setting $g(\pm\vec{e}) = \pm g(e)$ for $\vec{e} \in \omega$.

The *incidence matrix* of (Σ, ω) is a matrix $\boldsymbol{M} = \boldsymbol{M}(\Sigma, \omega)$ indexed by $V \times E$, whose entry at $(u, e) \in V \times E$ is

$$\boldsymbol{m}(u, e) = \begin{cases} [u, \vec{e}] & \text{if } e \text{ is a link, } \vec{e} \in \omega, \\ 2[u, \vec{e}] & \text{if } e \text{ is a negative loop, } \vec{e} \in \omega, \\ 0 & \text{otherwise.} \end{cases} \tag{3.2}$$

Alternatively, $\boldsymbol{m}(u, e) = \sum_{w \in \mathrm{End}(e),\, w = u} [w, \vec{e}]$ with $\vec{e} \in \omega$.

The matrix $\boldsymbol{M}$ is the matrix of the co-boundary operator δ. Let $\boldsymbol{m}_v$ denote the v-th row of $\boldsymbol{M}$ at $v \in V$. Then $\boldsymbol{m}_v$ is a function on E, and can be also viewed as a 1-chain of Σ given by $\boldsymbol{m}_v(\vec{e}) = \boldsymbol{m}_v(e)$ for $\vec{e} \in \omega$. Let 1_v denote the *delta potential* of Σ at $v \in V$, i.e., $1_v(u) = 1$ for $u = v$ and $1_v(u) = 0$ for $u \ne v$. We may write 1_v as a 0-chain v of Σ. Then $\delta v = \delta 1_v = \boldsymbol{m}_v$. The following proposition corrects Lemma 4.5(b) of Chen and Wang [4], expressing $\boldsymbol{m}_v$ as a sum of a possible semi-bond vector and some bond vectors of Type I.

Proposition 3.1. *Fix a vertex v and partition $\Sigma(V \smallsetminus v)$ into balanced components Σ_i, Σ'_j and unbalanced components Σ''_k, such that $\Sigma_i \cup \Sigma[v, \Sigma_i]$ are balanced and $\Sigma'_j \cup \Sigma[v, \Sigma'_j]$ are unbalanced. Then $U = [v, V \smallsetminus v] \cup E_v^-$ is a uni-cut, $B' = [v \cup X, Y \cup Z] \cup E_v^-$ is a semi-bond, and $B_i = [v, X_i]$ are bonds of Type I, where E_v^- is the set of negative loops at v,*

$$X_i = V(\Sigma_i), \quad X = \bigcup_i X_i, \quad Y = \bigcup_j V(\Sigma'_j), \quad Z = \bigcup_k V(\Sigma''_k).$$

Furthermore, given an orientation ω of Σ and a direction ω_U of U such that $[v, e] = 1$ for $e \in \omega_U$. Let $\boldsymbol{m}_v$ be the v-th row vector of the matrix $\boldsymbol{M}(\Sigma, \omega)$, and ν a switching function such that $\nu|_{Z \cup v} = 1$ and $\sigma^\nu|_{[v,X] \cup E(X \cup Y)} = 1$. Then

$$\begin{aligned} \boldsymbol{m}_v &= \delta v = I_{(U,\,\omega_U)} = I_{(B',\,\omega_U)} + \sum_i I_{(B_i,\,\omega_U)} && (3.3) \\ &= \frac{1}{2}\big[I_{(B'_1,\,\omega_U)} + I_{(B'_2,\,\omega_U)}\big] + \sum_i I_{(B_i,\,\omega_U)}, && (3.4) \end{aligned}$$

where $B'_1 = [Z^c, Z] \cup E_v^- \cup E^-\big(\Sigma^\nu[v, Y]\big)$ and $B'_2 = [Z^c, Z] \cup E_v^- \cup E^+\big(\Sigma^\nu[v, Y]\big)$.

Proof. The bonds B_i can be written as $B_i = [X_i, {X_i}^c]$ and are of Type I clearly. Note that all edges of $\Sigma^\nu[X_i, v]$ are positive. Since $\nu(v) = 1$, we still have $[v, e^\nu] = 1$ for $e \in \omega_U$. It follows that $[u, e^\nu] = -1$ for $e \in \omega_U$ with end-vertices $u \in X_i$. Thus the restriction $\omega_U|_{[v, X_i]}$ is a direction of B_i.

Notice that B' is a semi-bond of Σ and is decomposed into two bonds B'_1, B'_2, which overlap on $[v, Z]$. The two edge subsets $E^-\big(\Sigma^\nu[v, Y]\big)$, $E^+\big(\Sigma^\nu[v, Y]\big)$ are nonempty because of the unbalance of $\Sigma'_j \cup [v, \Sigma'_j]$. The restrictions of ω_U to B'_1, B'_2 are directions of B'_1, B'_2 respectively in Σ by Proposition 2.4(a). □

Each cut vector can be expressed in terms of the row vectors of the incidence matrix. It is was first stated for a bond vector in Lemma 4.5(a) of Chen and wang [4]. Here we modify it to a cut vector and simplify its original proof.

Proposition 3.2. *Let $U = [X, X^c] \cup E_X$ be a cut with a direction ω_U, and ν a switching function such that $\sigma^\nu|_{E_X} = -1$, $\sigma^\nu|_{E(X) \smallsetminus E_X} = 1$, and $[u, e^\nu] = 1$ for $e \in \omega_U$ with end-vertices $u \in X$. Then for each orientation ω of Σ,*

$$I_{(U,\,\omega_U)} = \sum_{v \in X} \nu(v) \boldsymbol{m}_v = \sum_{v \in X} \nu(v) \delta v. \tag{3.5}$$

Proof. We write $\omega = \{\vec{e} : e \in E\}$ and $\omega_U = \{\tilde{e} : e \in U\}$. For each edge $e \in U$ with an end-vertex $u \in X$ in Σ, the sign of $I_{(U,\,\omega_U)}(e) := I_{(U,\,\omega_U)}(\vec{e})$ is

$$[\vec{e}, \tilde{e}] = [\vec{e}^{\,\nu}, \tilde{e}^{\nu}] = [u, \vec{e}^{\,\nu}]\,[u, \tilde{e}^{\nu}] = [u, \vec{e}^{\,\nu}],$$

where $\vec{e} \in \omega$, $\tilde{e} \in \omega_U$, $[\vec{e}, \tilde{e}] = 1$ if $\vec{e} = \tilde{e}$ and $[\vec{e}, \tilde{e}] = -1$ if $\vec{e} = -\tilde{e}$. Then for each edge $e \in [X, X^c]$ with an end-vertex $u \in X$,

$$I_{(U,\,\omega_U)}(e) = [u, \vec{e}^{\,\nu}] = \sum_{w \in X} \boldsymbol{m}_w^\nu = \sum_{w \in X} \nu(w) \boldsymbol{m}_w;$$

and for each edge $e \in E_X$ with end-vertices $u, v \in X$,

$$I_{(U,\,\omega_U)}(e) = 2[u, \vec{e}^{\,\nu}] = [u, \vec{e}^{\,\nu}] + [v, \vec{e}^{\,\nu}] = \sum_{w \in X} \boldsymbol{m}_w^\nu = \sum_{w \in X} \nu(w) \boldsymbol{m}_w.$$

□

The following proposition is crucial in the proof of the Reduction Algorithm for Integral Tensions. We reproduce the proof here by using incidence numbers rather than coupling of orientations. A potential function p is said to be a *potential* of a tension g if $\delta p = g$.

Proposition 3.3 (Construction of Potential from Tension). *Let g be an integral tension of connected Σ.*

(a) *If Σ is balanced, then for each fixed vertex $u_0 \in V$ and $c \in \mathbb{Z}$ there exists a unique potential function p such that $\delta p = g$ with $p(u_0) = c$.*

(b) *If Σ is unbalanced, then there exists a unique potential function p such that $\delta p = g$. Moreover, if $p(u_0)$ is an odd half-integer, then $p(u)$ is an odd half-integer for all $u \in V$.*

Proof. Fix a vertex u_0 and assume the value $p(u_0)$. We define the value of p at an arbitrary vertex u as follows: Take a directed walk $\omega_W = u_0x_1u_1x_2 \ldots u_{n-1}x_nu_n$, where $u_n = u$, $x_i = u_{i-1}u_i$, and $[u_i, x_i][u_i, x_{i+1}] = -1$, $1 \le i \le n-1$. Since $\delta p = g$, whenever $p(u_{i-1})$ is given, the value $p(u_i)$ must be given by

$$(\delta p)(x_i) = [u_{i-1}, x_i]p(u_{i-1}) + [u_i, x_i]p(u_i) = g(x_i).$$

Since $[u_{i-1}, x_i][u_i, x_i] = -\sigma(x_i)$, we obtain the recurrence relation

$$p(u_i) = \sigma(x_i)p(u_{i-1}) + [u_i, x_i]g(x_i), \quad 1 \le i \le n. \tag{3.6}$$

Since ω_W is a directed walk, we have

$$\prod_{j=1}^{k} \sigma(x_j) = -[u_0, x_1][u_k, x_k], \quad 1 \le k \le n. \tag{3.7}$$

The recurrence relation (3.6) and (3.7) imply that

$$\begin{aligned} p(u_n) &= p(u_0)\prod_{i=1}^{n}\sigma(x_i) + \sum_{i=1}^{n}[u_i, x_i]g(x_i)\prod_{j=i+1}^{n}\sigma(x_j) \\ &= \left(p(u_0) + \sum_{i=1}^{n}[u_i, x_i]g(x_i)\prod_{j=1}^{i}\sigma(x_j)\right)\prod_{i=1}^{n}\sigma(x_i) \\ &= \left(p(u_0) - [u_0, x_1]\sum_{i=1}^{n}[u_i, x_i][u_i, x_i]g(x_i)\right)\prod_{i=1}^{n}\sigma(x_i). \end{aligned}$$

Thus, whenever $p(u_0)$ is given, the value $p(u)$ is determined by

$$p(u) = p(u_n) := \left[p(u_0) - [u_0, x_1]\sum_{i=1}^{n} g(x_i)\right]\prod_{i=1}^{n}\sigma(x_i). \tag{3.8}$$

We are left to show that p is well defined. It suffices to show that, when ω_W is a directed closed walk with $u_n = u_0$, we should have

$$p(u_0) = \left[p(u_0) - [u_0, x_1]\sum_{i=1}^{n} g(x_i)\right]\prod_{i=1}^{n}\sigma(x_i). \tag{3.9}$$

If W is a directed closed positive walk, then $\prod_{i=1}^{n}\sigma(x_i) = 1$ and $\sum_{i=1}^{n} g(x_i) = 0$ (as g is a tension). The identity (3.9) holds automatically.

(a) Σ *is balanced.* Since every closed walk of Σ has positive sign, the identity (3.8) holds for every directed closed walk. Then the value of p at the base vertex u_0 can be arbitrarily assigned. Set $p(u_0) = c$, the values of p at other vertices are uniquely determined.

(b) Σ *is unbalanced.* Let ω_W be a closed walk with negative sign, that is, $u_n = u_0$ and $\prod_{i=1}^{n} \sigma(x_i) = -1$. Then $p(u_0)$ is determined by (3.8) as follows:

$$p(u_0) = p(u_n) = \frac{[u_0, x_1]}{2} \sum_{i=1}^{n} g(x_i). \tag{3.10}$$

Next we show that $p(u_0)$ does not depend on the selection of the directed closed walk ω_W at u_0.

Let $\omega_{W'}$ be another directed closed walk at u_0 with negative sign, having the vertex-edge sequence $v_0 y_0 v_1 y_1 \ldots y_{m-1} v_m$, where $v_0 = v_m = u_0$. Let p' be defined by (3.8) with $\omega_{W'}$. It suffices to show that $p(u_0) = p'(v_0)$, where

$$p'(v_0) = p'(v_m) = \frac{[v_0, y_1]}{2} \sum_{i=1}^{m} g(y_i).$$

Let $[v_0, y_1] = -\theta[u_n, x_n]$ with $\theta \in \{-1, 1\}$. Since both ω_W and $\omega_{W'}$ are negative directed closed walks, we have $[u_n, x_n] = [u_0, x_1]$ and $[v_m, y_m] = [v_0, y_1]$. Then $[v_0, y_1] = -\theta[u_0, x_1]$, and the concatenation $\omega_{WW'}$ is a directed closed walk with positive sign, whose oriented vertex-edge sequence is

$$u_0 x_1 u_1 x_2 \ldots u_{n-1} x_n u_n (v_0)(\theta y_1) v_1 (\theta y_2) \ldots v_{m-1} (\theta y_m) v_m.$$

Since g is a tension, then

$$\langle \omega_{WW'}, g \rangle = \sum_{i=1}^{n} g(x_i) + \sum_{j=1}^{m} g(\theta y_j) = \sum_{i=1}^{n} g(x_i) + \theta \sum_{j=1}^{m} g(y_j) = 0.$$

Hence

$$p(u_0) = \frac{[u_0, x_1]}{2} \sum_{i=1}^{n} g(x_i) = -\frac{\theta[v_0, y_1]}{2} \sum_{i=1}^{m} g(y_i) = p'(v_0).$$

The odd half-integer property of p follows from (3.6). $\square$

It is well known that the three lattices $B(G, \mathbb{Z})$, $\mathrm{Row}_{\mathbb{Z}} \boldsymbol{M}(G)$, $T(G, \mathbb{Z})$ are identical for a unsigned graph $G = (V, E)$. Since every balanced signed graph is equivalent to a unsigned graph by switching, it follows that the three lattices are identical for balanced signed graphs. The following proposition combines Theorem 4.6 and Corollary 4.10 of Chen and Wang [4], and adds a new relationship between $B(G, \mathbb{Z})$ and $\tilde{B}(G, \mathbb{Z})$ when Σ is unbalanced. The proof here corrects some errors in their original proof.

Proposition 3.4. (a) $Z(\Sigma, \mathbb{Z}) = F(\Sigma, \mathbb{Z})$, $\tilde{B}(\Sigma, \mathbb{Z}) = T(\Sigma, \mathbb{Z})$.

(b) $\delta C_0(\Sigma, \mathbb{Z}) = \mathrm{Row}_{\mathbb{Z}} \boldsymbol{M}(\Sigma, \omega) = B(\Sigma, \mathbb{Z})$, *and is the* $\mathbb{Z}$*-span of cut vectors of* Σ.

(c) *If* Σ *is balanced, then* $B(\Sigma, \mathbb{Z}) = \tilde{B}(\Sigma, \mathbb{Z})$.

(d) *If* Σ *is connected and unbalanced, then* $\tilde{B}(\Sigma, \mathbb{Z}) = B(\Sigma, \mathbb{Z}) \cup (h + B(\Sigma, \mathbb{Z}))$ *(disjoint),*

$$\delta^{-1} \tilde{B}(\Sigma, \mathbb{Z}) = C_0(\Sigma, \mathbb{Z}) \cup (p + C_0(\Sigma, \mathbb{Z})) \text{ (disjoint)}, \tag{3.11}$$

where h *is a hyper-bond vector and* p *is a potential function having odd half-integer value everywhere.*

Proof. (a) It follows from Theorem 4.9 of Chen and Wang [4]. The second identity follows also from the Reduction Algorithm of Section 4, Proposition 2.4(b), and Proposition 2.5(b).

(b) The first identity is by definition. Notice that the equation (3.3) implies that $\mathrm{Row}_{\mathbb{Z}} \boldsymbol{M}(\Sigma, \omega)$ is contained in the $\mathbb{Z}$-span of cut vectors, and that the equation (3.5) implies that the $\mathbb{Z}$-span of cut vectors is contained in $\mathrm{Row}_{\mathbb{Z}} \boldsymbol{M}(\Sigma, \omega)$. So $\mathrm{Row}_{\mathbb{Z}} \boldsymbol{M}(\Sigma, \omega)$ is the $\mathbb{Z}$-span of cut vectors. Propositions 2.3(c) implies the $\mathbb{Z}$-span of cut vectors are contained in $B(\Sigma, \mathbb{Z})$. Proposition 3.1 implies that $\mathrm{Row}_{\mathbb{Z}} \boldsymbol{M}(\Sigma, \omega) \subset B(\Sigma, \mathbb{Z})$. Proposition 3.2 implies that $B(\Sigma, \mathbb{Z}) \subset \mathrm{Row}_{\mathbb{Z}} \boldsymbol{M}(\Sigma, \omega)$. Hence $\mathrm{Row}_{\mathbb{Z}} \boldsymbol{M}(\Sigma, \omega) = B(\Sigma, \mathbb{Z})$.

(c) Trivially equivalent to graph case.

(d) The first identity follows from the fact that the potential function of any hyper-bond vector h has values $\pm\frac{1}{2}$ everywhere. The identity (3.11) follows from Proposition 3.3(b). $\square$

4 Conformal decomposition of integral tensions and integral potentials

Given an integral potential function p of connected Σ. The vertex set V is partitioned into sets $p^{-1}(a)$ of potentials $a \in \mathbb{Z}$. Each connected component of $\Sigma(p^{-1}(a))$ is called an *equal-potential component* of Σ with potential a or *ep-component* for short.

Lemma 4.1. *Let g be a nonzero integral tension of a connected Σ, and p a potential function such that $\delta p = g$. Then ep-components of Σ with respect to p satisfy the following properties:*

(a) g is zero on all edges inside each ep-component of potential 0.

(b) g is zero on all positive edges and nonzero on all negative edges inside each ep-component of nonzero potential.

(c) If two ep-components of non-opposite potentials are connected by some edges, that is, not the case that one has potential $a > 0$ and the other has potential $-a$, then g is nonzero on the edges between the two ep-components.

(d) If there exists an ep-component of odd half-integer potential, then all ep-components have odd half-integer potentials.

Proof. All follow from $(\delta p)(e) = [u,e]p(u) + [v,e]p(v)$, where $e \in \vec{E}$ with $\mathrm{End}(e) = \{u,v\}$. □

Let $U = [X, X^c] \cup E_X$ be a cut of Σ with a direction ω_U. Let ν be a switching such that $\nu|_{X^c} = 1$ and $[v, e^\nu] = 1$ for $e \in \omega_U$ with end-vertices $v \in X$. We define a potential function $p_{(U,\,\omega_U)} : V \to \mathbb{Z}$ by

$$p_{(U,\,\omega_U)}(v) = \begin{cases} [v,e] & \text{if} \quad v \in X, \\ 0 & \text{if} \quad v \notin X, \end{cases} \tag{4.1}$$

where $e \in \omega_U$ with $v \in \mathrm{End}(e)$. Then $p^\nu_{(U,\,\omega_U)}(v) = 1$ for $v \in X$ and $p^\nu_{(U,\,\omega_U)}(v) = 0$ for $v \notin X$. By definition of co-boundary operator δ, we see that

$$\delta p_{(U,\,\omega_U)} = I_{(U,\,\omega_U)}. \tag{4.2}$$

Theorem 4.2 (Reduction Algorithm for Integral Tensions)**.** *Let g be a nonzero integral tension of connected Σ. Then g can be decomposed conformally into a sum of some bond vectors of Types I or III, some semi-bond vectors, and a possible hyper-bond vector.*

Proof. `STEP 1`: Input the tension g and find its potential function p by Proposition 3.3. Whenever Σ is balanced, make p to be integer-valued. Let $S = \{v \in V \,|\, p(v) < 0\}$, and ν be a switching given by $\nu(v) = -1$ for $v \in S$ and $\nu(v) = 1$ for $v \notin S$. Then $p^\nu \geq 0$, g^ν is a nonzero integral tension of Σ^ν, and $\delta p^\nu = g^\nu$. Partition Σ^ν into equal-potential components by the potential function p^ν.

`STEP 2`: *There are two ep-components of distinct potentials in Σ^ν.*

We select an ep-component Σ^ν_a of highest potential a. Then each ep-component Σ^ν_b that connects Σ^ν_a by some edges must have potential $b < a$. Note that $a - b \geq 1$ by Lemma 4.1(d) so that $a \geq 1$. Remove the negative edge set $E^-(\Sigma^\nu_a)$ from Σ^ν_a and select a balanced component Σ^0_a of $\Sigma^\nu_a \smallsetminus E^-(\Sigma^\nu_a)$. Set $X := V(\Sigma^0_a)$ and let $E_X \subset E(X)$ be the negative edge set of $\Sigma(X)$. Then

$$U := [X, X^c] \cup E_X$$

is a uni-cut of Types I or III for Σ and $[X, X^c] \neq \varnothing$. Let ω_U be a direction of U such that $[u, e^\nu] = 1$ for $e \in \omega_U$ with end-vertices $u \in X$. Then $g(e) = g^\nu(e^\nu) = 2a \geq 2$ for $e \in \omega_U$ with end-vertices $u, v \in X$, and

$$g(e) = g^\nu(e^\nu) = p^\nu(u)[u,e^\nu] + p^\nu(v)[v,e^\nu] = a \pm b \geq 1$$

for $e = uv \in \omega_U$ with $u \in X, v \in V(\Sigma^\nu_b)$. It follows that $I_{(U,\,\omega_U)}$ conforms to the sign pattern of g and $I_{(U,\,\omega_U)} \leq g$ within ω_U. Let $p_{(U,\,\omega_U)}$ be the potential function of $I_{(U,\,\omega_U)}$ defined by (4.1).

Then $g' := g - I_{(U,\omega_U)}$ is an integral tension of Σ with potential function $p' := p - p_{(U,\omega_U)}$. If $g' = 0$, STOP (g is conformally decomposed into uni-cut vectors of Types I and III in Σ). If $g' \neq 0$, write g' as g and p' as p, return to STEP 1. Otherwise, Σ^ν has constant potential, then go to STEP 3.

STEP 3: Σ^ν *has constant potential* $a \geq 1$.

Since Σ^ν is connected, there is only one ep-component and it must be Σ^ν. Since g^ν is nonzero, Σ^ν must contain some negative edges. Remove the negative edge set $E^-(\Sigma^\nu)$ from Σ^ν and select a component Σ^0 of $\Sigma^\nu \setminus E^-(\Sigma^\nu)$, which must be balanced. Set $X := V(\Sigma^0)$ and let $E_X \subset E(X)$ be the negative edge set of $\Sigma(X)$. Then $U := [X, X^c] \cup E_X$ is a uni-cut of Types I or III for Σ if $\Sigma^\nu \setminus E^-(\Sigma^\nu)$ is disconnected, and U is a bond of Type II for Σ if $\Sigma^\nu \setminus E^-(\Sigma^\nu)$ is connected. Let ω_U be a direction of U such that $[u, e^\nu] = 1$ for $e \in \omega_U$ with end-vertices $u \in X$. Since $a \geq 1$, then $g(e) = g^\nu(e^\nu) = 2a \geq 2$ for all $e = uv \in \omega_U$ with $u, v \in X$. It follows analogously that $I_{(U,\omega_U)}$ conforms to the sign pattern of g and $I_{(U,\omega_U)} \leq g$ within ω_U. Let $p_{(U,\omega_U)}$ be the potential function of $I_{(U,\omega_U)}$ defined by (4.1). Then $g' := g - I_{(U,\omega_U)}$ is an integral tension of Σ with potential function $p' := p - p_{(U,\omega_U)}$. If $g' = 0$, STOP (g is conformally decomposed into uni-cut vectors of Types I and III in Σ). If $g' \neq 0$, write g' as g and p' as p, return to STEP 1. Otherwise, Σ^ν has constant potential $\frac{1}{2}$, then go to STEP 4.

STEP 4: Σ^ν *has constant potential* $\frac{1}{2}$.

Since g is nonzero, Σ must contain some negative edges. Remove the negative edge set $E^-(\Sigma^\nu)$ from Σ^ν. Let $\Sigma^\nu_1, \ldots, \Sigma^\nu_m$ denote the components of $\Sigma^\nu \setminus E^-(\Sigma^\nu)$, which consists of only positive edges of Σ^ν. If $m = 1$, then $E_{X_1} \neq \varnothing$. Set $X_i := V(\Sigma^\nu_i)$ and let E_{X_i} be negative edge set of $\Sigma(X_i)$. Define

$$U := \bigcup_{i<j} [X_i, X_j] \cup \bigcup_{k=1}^{m} E_{X_k}.$$

Then $U = E^-(\Sigma^\nu)$. Let ω be an orientation of Σ such that $\omega_U := \omega|_U$ is a direction of U and $[u, e^\nu] = 1$ for all $e \in \omega_U$ with $u \in \mathrm{End}(e)$. We then have

$$g(e) = \begin{cases} 1 & \text{if } e \in \omega_U, \\ 0 & \text{if } e \in \omega \setminus \omega_U \cup (-\omega_U). \end{cases}$$

If $m \geq 2$ and $E_{X_i} = \varnothing$ for at least one i, then $U_i = [X_i, {X_i}^c]$ is a uni-cut of Type I for Σ, and $\omega_i := \omega|_{U_i}$ is a direction of U_i. The uni-cut vector $I_{(U_i,\omega_i)}$ confirms to the sign pattern of g and $I_{(U_i,\omega_i)} \leq g$ within ω_U. Let $p_{(U,\omega_U)}$ be the potential function of $I_{(U,\omega_U)}$ defined by (4.1). Then $g' := g - I_{(U,\omega_U)}$ is a tension of Σ with potential function $p' := p - p_{(U_i,\omega_i)}$. However, $p'(v) = -\frac{1}{2}p(v)$ for $v \in X_i$ and $p'(v) = p(v)$ for $v \notin X_i$. If $g' = 0$, STOP (g is conformally decomposed into uni-cut vectors of Σ). If $g' \neq 0$, write g' as g and p' as p, return to STEP 1. Otherwise, $E_{X_i} \neq \varnothing$ for all i. Then U is a hyper-bond of Σ. We obtain a hyper-bond vector $I_{(U,\omega_U)}$, where $I_{(U,\omega_U)}(e) = 1$ for $e \in \omega_U$ and $I_{(U,\omega_U)}(e) = 0$ for $e \in \omega \setminus \omega_U \cup (-\omega_U)$. It follows that $g = I_{(U,\omega_U)}$. STOP (g is conformally decomposed into a hyper-bond vector and some uni-cut vectors).

Now recall Proposition 2.3(b) that each uni-cut vector of Types I and III can be decomposed conformally into some bond vectors of Type I, plus a bond vector of Type III or a semi-bond vector. It follows that g is decomposed conformally into a sum of bond vectors of Types I and III, some semi-bond vectors, and a possible hyper-bond vector. □

It is well-known that a nonzero integral tension of an ordinary graph may have several conformal decompositions. Nonzero integral tensions of unbalanced connected signed graphs may also have several conformal decompositions. Figure 8 is a cut vector which can be conformally decomposed into conformally indecomposable integral tensions in three different ways. Figure 9 demonstrates such three decompositions in three rows: the first row consists of two bond vectors of Type III, the second row consists of two bond vectors of Types II and III, and the last row consists of two hyper-bond vectors.

Recall that an integral potential function $p : V \to \mathbb{Z}$ is said to be *conformally decomposable* if there exist nonzero integral potential functions p_1, p_2 such that $p = p_1 + p_2$, $p_1(v)p_2(v) \geq 0$

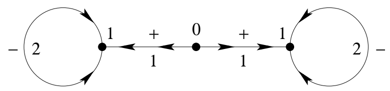

Figure 8: A cut vector (characteristic vector of a directed cut)

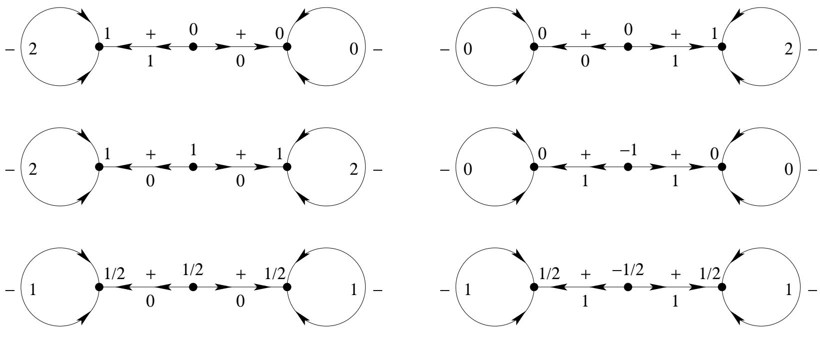

Figure 9: Three conformal decompositions

for all $v \in V$ and $\delta p_1(e)\delta p_2(e) \geq 0$ for all $e \in \vec{E}$. Notice that $p_1(v)p_2(v) \geq 0$ for $v \in V$ does not imply that $\delta p = \delta p_1 + \delta p_2$ is a conformal decomposition. An integral potential function is said to be *conformally indecomposable* if it is *not* conformally decomposable.

Corollary 4.3. *Let p be a nonzero integral potential of Σ. Then p is conformally indecomposable if and only if there exists a directed uni-cut (U, ω_U) of Σ such that $p = p_{(U,\,\omega_U)}$.*

Proof. We may assume that Σ is connected. Since p is integral (not half-integer valued), the integral tension δp can be conformally decomposed into uni-cut vectors by the Tension Reduction Algorithm (up to `STEP 2`), say, $\delta p = \sum_{i=1}^{m} I_{(U_i,\,\omega_i)}$, where (U_i, ω_i) are directed uni-cuts of Σ. Then $p = \sum_{i=1}^{m} p_{(U_i,\,\omega_i)}$ is a conformal decomposition, where $\delta p_{(U_i,\,\omega_i)} = I_{(U_i,\,\omega_i)}$. If p is conformally indecomposable, we must have $m = 1$ and δp is a uni-cut vector.

Conversely, let $p = p_{(U,\,\omega)}$ for a uni-cut $U = [X, X^c] \cup E_X$ of Σ with a direction ω_U. Suppose p is conformally decomposed into $p = p_1 + p_2$, then δp is conformally decomposed into $\delta p = \delta p_1 + \delta p_2$. Let ν be a switching such that $\nu|_{X^c} = 1$, $\sigma^\nu|_{E_X} = -1$, $\sigma^\nu|_{E(X) \smallsetminus E_X} = 1$, and $[u, e^\nu] = 1$ for $e \in \omega_U$ with end-vertices $u \in X$. Then $p^\nu(v) = 1$ for $v \in X$ and $p^\nu(v) = 0$ otherwise. Moreover, $p^\nu = p_1^\nu + p_2^\nu$ and $\delta p^\nu = \delta p_1^\nu + \delta p_2^\nu$. The conformal decomposition $p = p_1 + p_2$ implies that X is partitioned into disjoint nonempty subsets X_1, X_2 such that $\operatorname{supp} p_1^\nu = X_1$, $\operatorname{supp} p_2^\nu = X_2$, $p_1^\nu|_{X_1} = 1$, $p_2^\nu|_{X_2} = 1$. Let $e \in E(X) \smallsetminus E_X$ be a positive edge in Σ^ν with end-vertices $u \in X_1, v \in X_2$. Then $\delta p_1^\nu(\vec{e}) = -\delta p_2^\nu(\vec{e}) = \pm 1$, where $\vec{e}$ is the edge e with an orientation. This is contradictory to the conformal decomposition $\delta p^\nu = \delta p_1^\nu + \delta p_2^\nu$. □

Proof of Main Theorem and Corollaries 1 and 2.

Proof. It is known from Propositions 2.7, 2.8, and 2.9 that reduced bond vectors, semi-bond vectors, and hyper-bond vectors are conformally indecomposable. Theorem 4.2 implies that indecomposable integral tensions are either bond vectors of Types I or III, or semi-bond vectors, or hyper-bond vectors. This finishes the proof of Main Theorem.

We have seen from Proposition 2.7 that the support of every bond vector of Σ is a circuit of $M^*(\Sigma, \mathbb{Z})$. Let g be a nonzero integral tension of Σ such that $\operatorname{supp} g$ is minimal. Let g be decomposed conformally into $g = \sum g_i$, where g_i are nonzero reduced bond vectors, or semi-bond vectors, or hyper-bond vectors. Since the support of any semi-bond vector or any hyper-bond vector contains sub-bond properly, it follows that g_i must be reduced bond vectors, $\operatorname{supp} g_i$ are the same bond, and $\operatorname{supp} g_i = \operatorname{supp} g$. This means that every circuit of $M^*(\Sigma, \mathbb{Z})$ is a bond of Σ.

Corollary 2 is equivalent to Corollary 4.3, where $U = [X, X^c] \cup E_X$. □

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